Quadratic Maximum-Weight Independent Set Problems (Q-MWIS)

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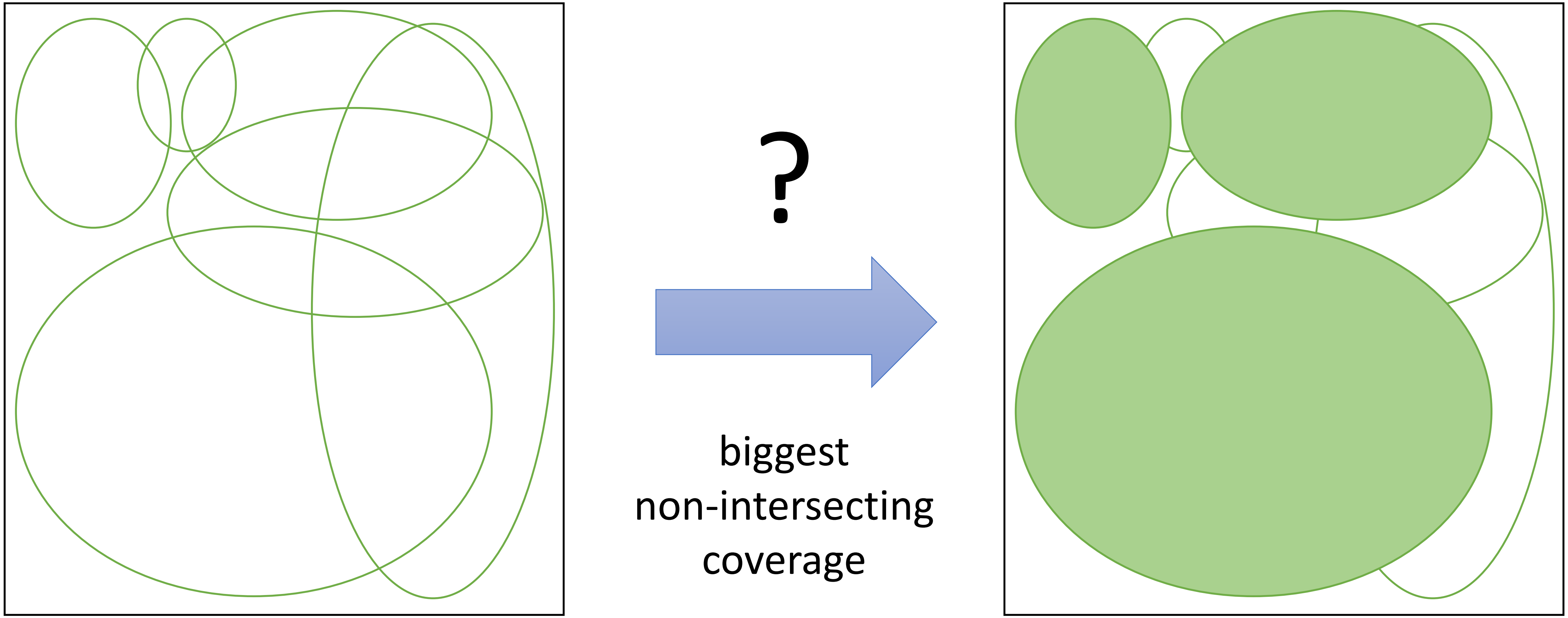
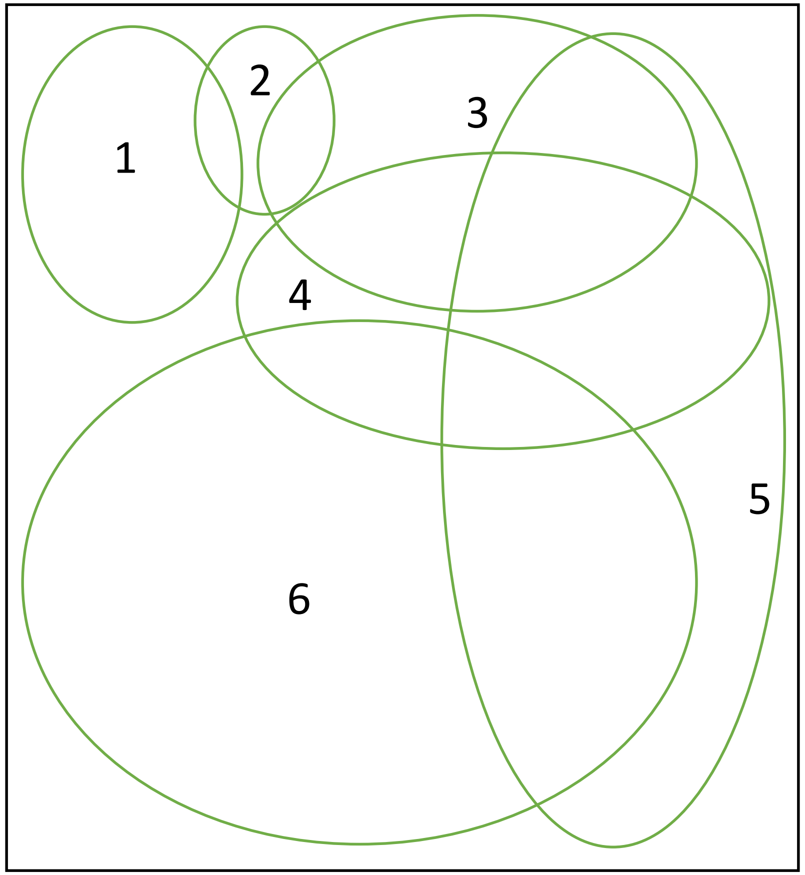
Example: Segmentation



?

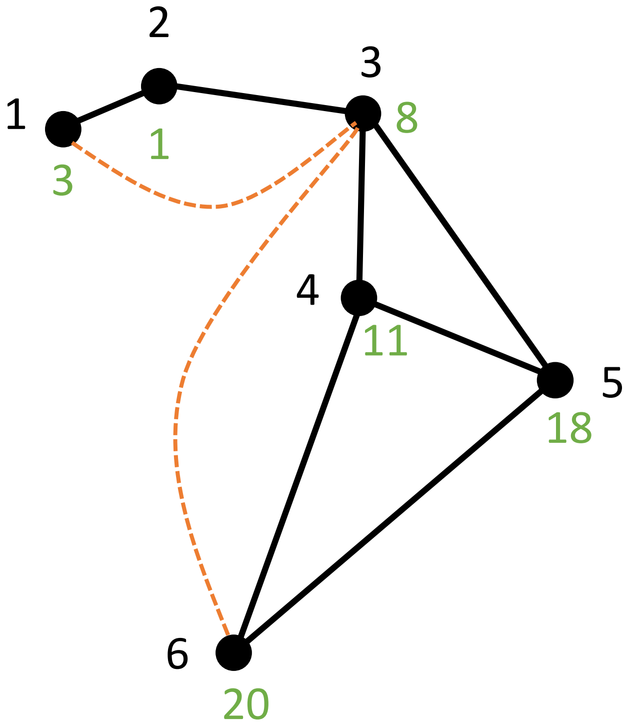
biggest
non-intersecting
coverage

Example: Segmentation

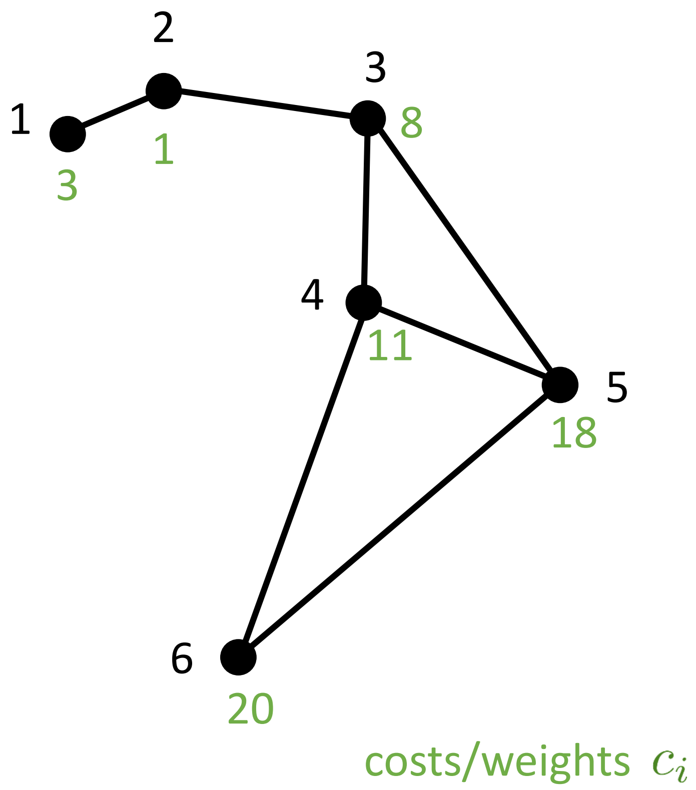


maximum weight
independent set
(MWIS)

costs/weights c_i



Example: Segmentation



integer linear program
(ILP)

$$\max_{x\in\{0,1\}^6} \sum_{i=1}^{6} c_i x_i$$

$$\begin{aligned} \text{s.t.}\quad & x_1 + x_2 \leq 1 \\ & x_2 + x_3 \leq 1 \\ & x_3 + x_4 + x_5 \leq 1 \\ & x_4 + x_5 + x_6 \leq 1 \end{aligned}$$

NP-hard, solvable with

- branch and bound [1]
- local search heuristics [2]
- ILP solvers (e.g. Gurobi)

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[1] Lamm et al., “Exactly solving the maximum weight independent set problem on large real-world graphs” in Proceedings ALENEX 2019, pp. 144–158, SIAM
[2] Dong et al., “A Local Search Algorithm for Large Maximum Weight Independent Set Problems,” in Proceedings ESA 2022, vol. 244 LIPIcs, pp. 45:1–45:16

Quadratic Maximum-Weight Independent Set

Some problems require modeling pairwise (quadratic) relations between labels…

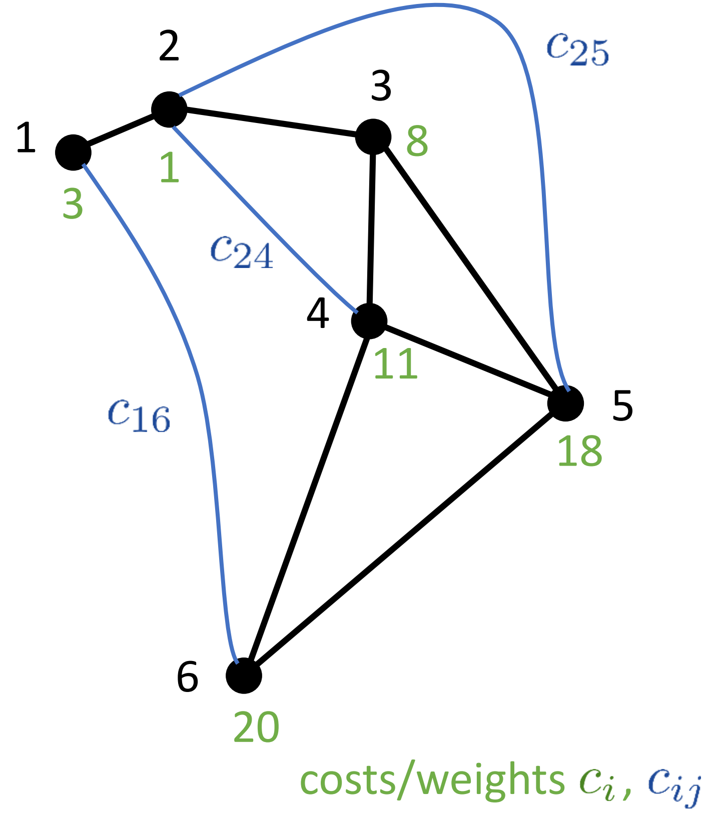
$$\max_{x \in \{0,1\}^6} \sum_{i=1}^{6} c_i x_i + c_{16} x_1 x_6 + c_{24} x_2 x_4 + c_{25} x_2 x_5$$

$$\text{s.t. } x_1 + x_2 \leq 1$$

$$\dots$$

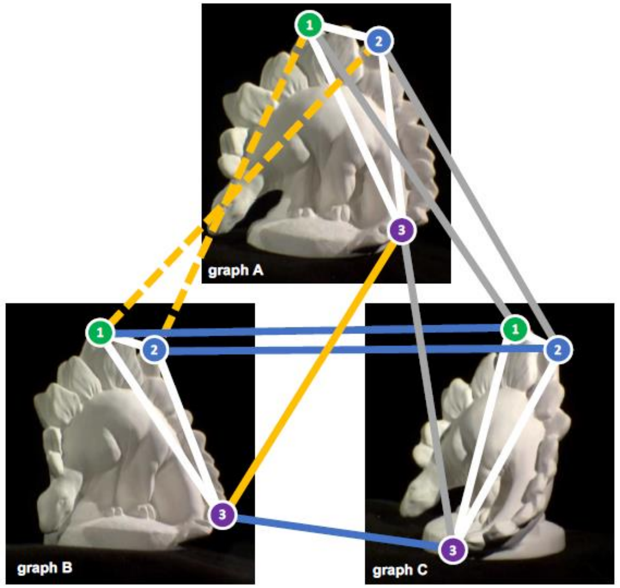
=> Quadratic Maximum-Weight Independent Set Problem (Q-MWIS)
NP-hard, solvable with…

?



Relation to similar problems

(Multi-)graph matching [3]

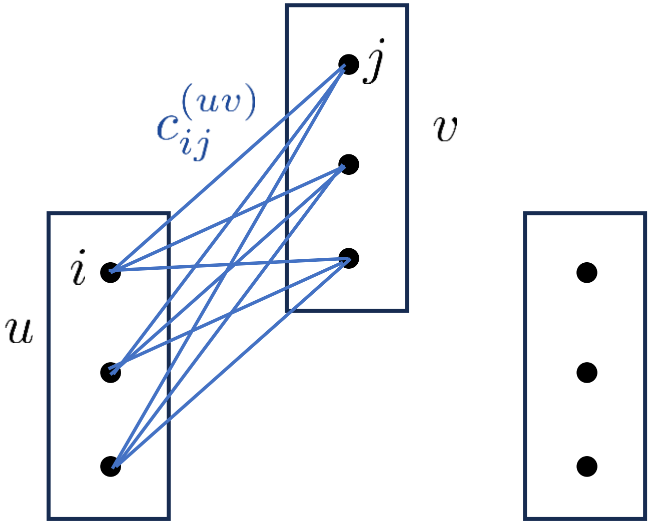


$$\min_{x\in\{0,1\}^{\mathcal{V}\times\mathcal{L}}} \sum_{i,j\in\mathcal{V}} \sum_{s,l\in\mathcal{L}} c_{is,jl} x_{is} x_{jl}$$

$$\text{s.t.} \begin{cases} \forall i\in\mathcal{V}: & \sum_{s\in\mathcal{L}} x_{is} \leq 1 \text{ and} \\ \forall s\in\mathcal{L}: & \sum_{i\in\mathcal{V}} x_{is} \leq 1. \end{cases}$$

[4] ≈ QAP

Maximum a posteriori (MAP) inference for graphical models



$$\min_{x\in\{0,1\}^N} \sum_{u\in\mathcal{V}} \sum_{i\in\mathcal{Y}_u} c_i^{(u)} x_i^{(u)} + \sum_{uv\in\mathcal{E}} \sum_{i\in\mathcal{Y}_u} \sum_{j\in\mathcal{Y}_v} c_{ij}^{(uv)} x_i^{(u)} x_j^{(v)},$$

$$\text{s.t. } \forall u\in\mathcal{V}: \sum_{i\in\mathcal{Y}_u} x_i^{(u)} = 1.$$

[3] P. Swoboda et al., “A convex relaxation for multi-graph matching,” in Proceedings of the IEEE/CVF Conference, pp. 11156–11165, 2019

[4] S. Haller et al., “A Comparative Study of Graph Matching Algorithms in Computer Vision”, ECCV 2022

Goals of the thesis

1. Introduce the Quadratic Maximum-Weight Independent Set Problem (Q-MWIS)
2. Construction of an „efficient“ linearization

quadratic integer program (QIP)

$$\max_{x\in\{0,1\}^6} \sum_{i=1}^{6} c_i x_i + c_{16}x_1x_6 + c_{24}x_2x_4 + c_{25}x_2x_5$$

$$\text{s.t. } x_1 + x_2 \leq 1$$

$$\dots$$

integer linear program (ILP)

$$\max_{\substack{x\in\{0,1\}^6\\ y\in\{0,1\}^3}} \sum_{i=1}^{6} c_i x_i + c_{16}y_{16} + c_{24}y_{24} + c_{25}y_{25}$$

$$\text{s.t. } x_1 + x_2 \leq 1$$

$$\dots$$

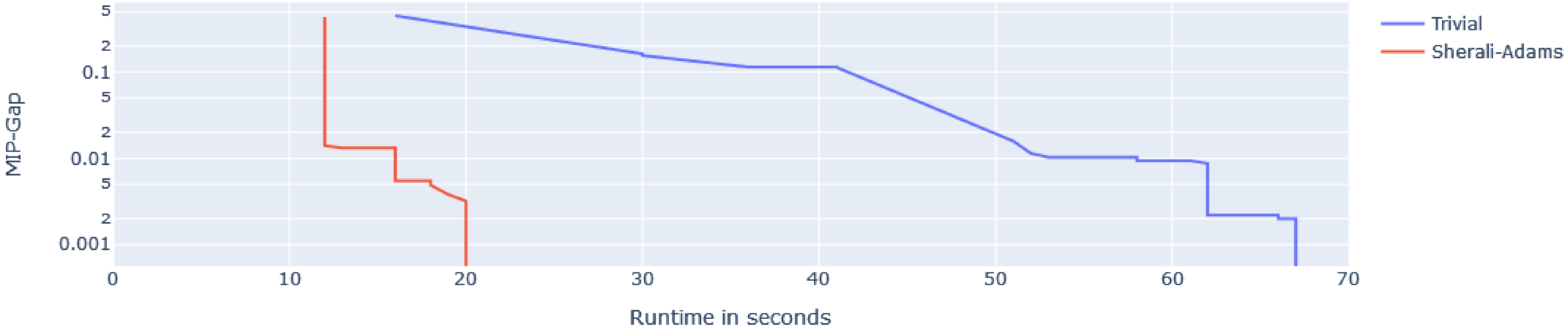
Linearization methods

„trivial“ linearization:

replace $x_i x_k := y_{ik}$

add constraints
$$y_{ik} \leq x_i,$$
$$y_{ik} \leq x_k,$$
$$y_{ik} \geq x_i + x_k - 1.$$

My main work: Linearization according to Sherali-Adams [5]

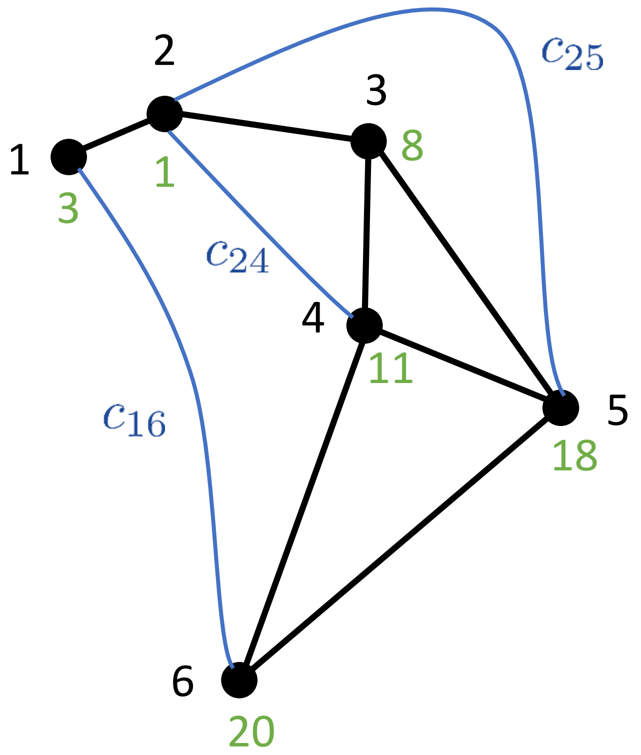


[5] Sherali et al., “A hierarchy of relaxations between the continuous and convex hull representations for zero-one programming problems,” SIAM Journal on Discrete Mathematics, vol. 3, no. 3, pp. 411–430, 1990

Outline

1. Q-MWIS problem formulation
2. Sherali-Adams linearization
3. Performance results of linearizations

General problem formulation



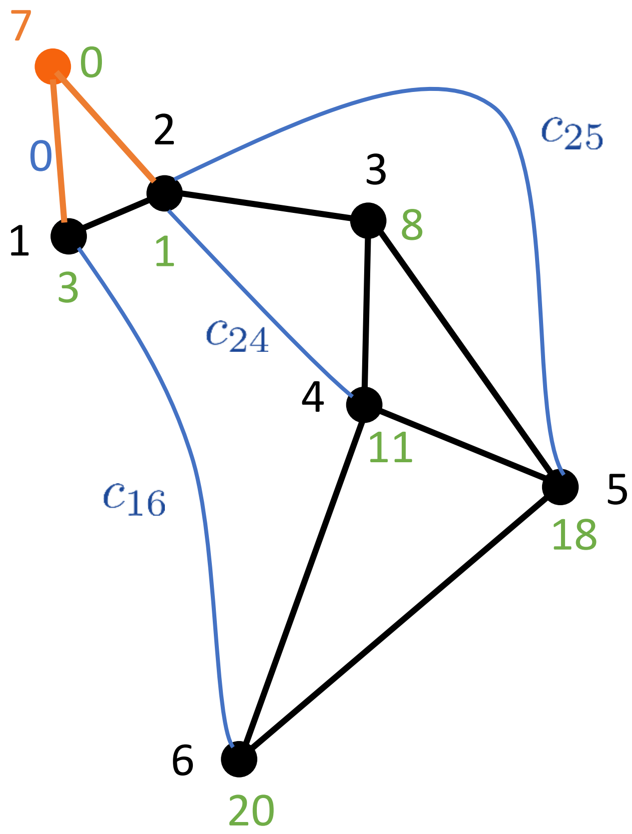
labels	$[n] := \{1, \ldots, n\}$
label pairs	$[[n]]^2 := \{ik \mid i, k \in [n], i < k\} \supseteq N_Z$
conflict/clique sets	$K_j \subseteq [n], \ \forall j \in [m]$
unary/pairwise costs	c_i, c_{ij}

$$\max_{x \in \{0,1\}^n} \sum_{i=1}^{n} c_i x_i + \sum_{ik \in N_Z} c_{ik} x_i x_k$$

$$\text{s.t.} \ \sum_{i \in K_j} x_i \leq 1, \ \forall j \in [m].$$

quadratic integer program (QIP)

Equality constraint reformulation

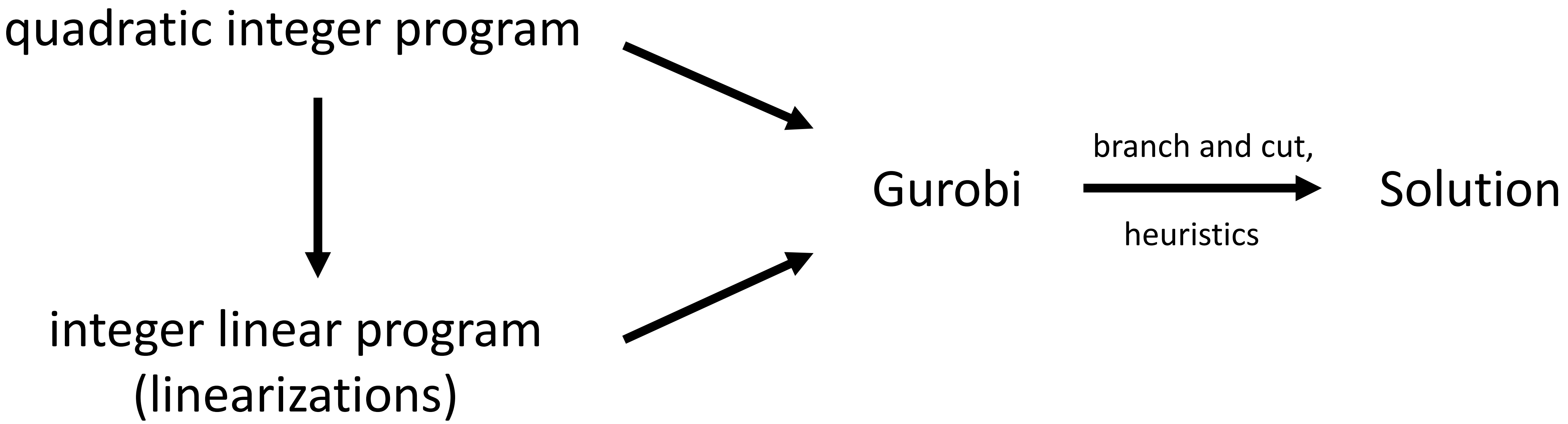


labels	$[n] = \{1, \dots, n\}$
slack labels	$\{n+1, \dots, n+m\}$
label pairs	$[[n+m]]^2$
new conflict/clique sets	$K'_j := K_j \cup \{j\} \subseteq [n+m], \ \forall j \in [m]$
unary/pairwise costs	c_i, c_{ij}

$$\max_{x \in \{0,1\}^{n+m}} \sum_{i=1}^{n} c_i x_i + \sum_{ik \in N_Z} c_{ik} x_i x_k$$

$$\text{s.t.} \sum_{i \in K'_j} x_i = 1, \ \forall j \in [m].$$

Solution approaches



quadratic integer program

integer linear program
(linearizations)

Gurobi

branch and cut,
heuristics

Solution

Trivial linearization

replace $$x_i x_k := y_{ik}$$

add constraints
$$y_{ik} \leq x_i,$$
$$y_{ik} \leq x_k,$$
$$y_{ik} \geq x_i + x_k - 1.$$

$$\max_{\substack{x \in \{0,1\}^{n+m} \\ y \in \{0,1\}^{|N_Z|}}} \sum_{i=1}^{n} c_i x_i + \sum_{ik \in N_Z} c_{ik} y_{ik}$$
$$\text{s.t.} \sum_{i \in K'_j} x_i = 1, \ \forall j \in [m],$$
$$y_{ik} \leq x_i, \ \forall ik \in N_Z,$$
$$y_{ik} \leq x_k, \ \forall ik \in N_Z,$$
$$y_{ik} \geq x_i + x_k - 1, \ \forall ik \in N_Z.$$

Role of constraints

$$\begin{aligned} \max_{x \in \{0,1\}^3} \sum_{i=1}^{3} c_i x_i, \\ \text{s.t. } x_1 + x_2 \leq 1, \\ x_2 + x_3 \leq 1, \\ x_1 + x_3 \leq 1. \end{aligned}$$

$$\xrightarrow{\text{Linear Programming (LP) relaxation}}$$

$$\begin{aligned} \max_{x \in [0,1]^3} \sum_{i=1}^{3} c_i x_i, \\ \text{s.t. } x_1 + x_2 \leq 1, \\ x_2 + x_3 \leq 1, \\ x_1 + x_3 \leq 1. \end{aligned}$$

or

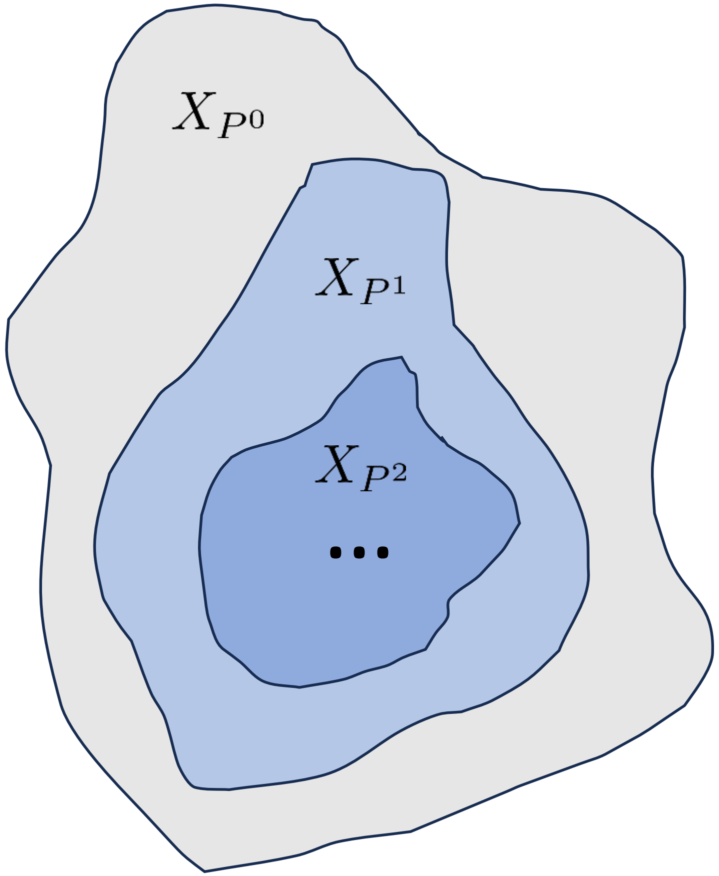
$$\neq$$

$$\begin{aligned} \max_{x \in \{0,1\}^3} \sum_{i=1}^{3} c_i x_i, \\ \text{s.t. } x_1 + x_2 + x_3 \leq 1. \end{aligned}$$

$$\longrightarrow$$

$$\begin{aligned} \max_{x \in [0,1]^3} \sum_{i=1}^{3} c_i x_i, \\ \text{s.t. } x_1 + x_2 + x_3 \leq 1. \end{aligned}$$

Sherali-Adams linearization



feasible set of relaxation

Progressively tighter LP relaxations…

… at the cost of more variables/constraints

=> First order linearization as middle ground

Sherali-Adams linearization of order d

Method:

1. Multiply all constraints with all polynomials

$$F_d(J_1, J_2) = \left(\prod_{j \in J_1} x_j\right)\left(\prod_{j \in J_2} (1 - x_j)\right), \text{ where}$$

$$J_1, J_2 \subseteq [n], \; J_1 \cap J_2 = \emptyset \text{ and } |J_1 \cup J_2| = d.$$

2. Add non-negativity constraints

$$F_{d+1}(J_1, J_2) \geq 0, \text{ for all such } J_1, \, J_2 \text{ of order } d+1.$$

3. Use relations on constraints:

$$x_i^2 = x_i, \text{ or equivalently } x_i(1 - x_i) = 0, \; \forall i \in [n].$$

4. Substitute remaining quadratic terms:

$$x_i x_j = w_{ij}$$

Sherali-Adams linearization Example

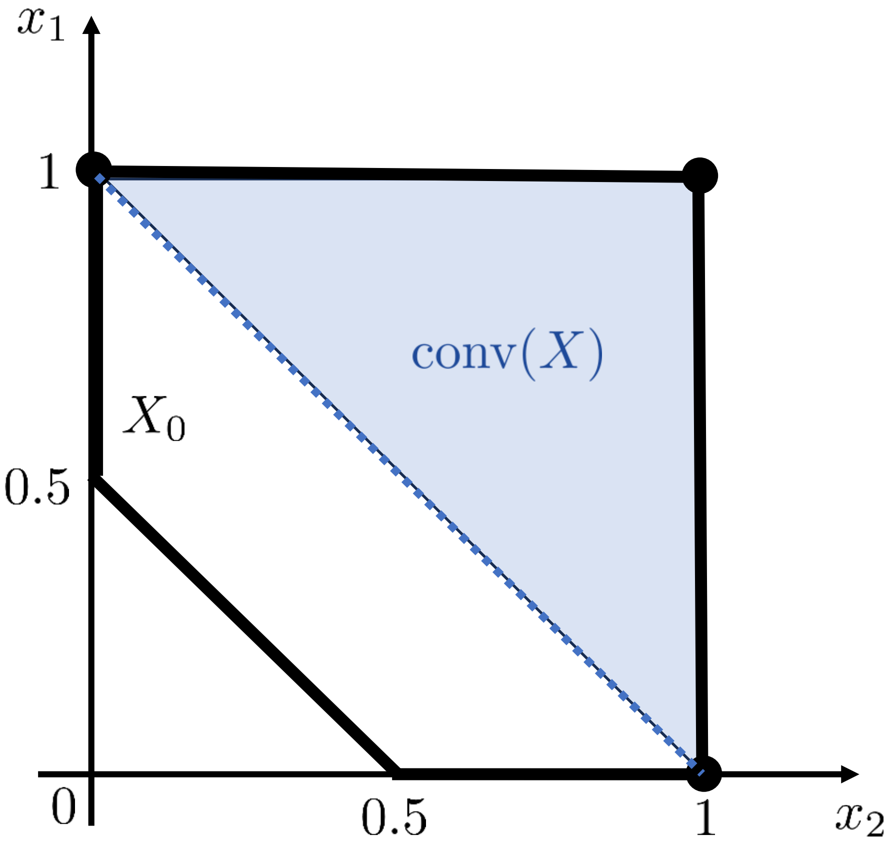
$$\max_{x \in X} \langle c, x \rangle$$

$$X = \left\{(x_1, x_2) \in \{0,1\}^2 \mid 2x_1 + 2x_2 \geq 1\right\}$$

$$= \{(0,1), (1,0), (1,1)\}$$

$$\mathrm{conv}(X) = \left\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 + x_2 \geq 1,\ x_1 \leq 1,\ x_2 \leq 1\right\}$$

$$X_{P^0} \equiv X_0 = \left\{(x_1, x_2) \in [0,1]^2 \mid 2x_1 + 2x_2 \geq 1\right\}$$



Sherali-Adams linearization Example

$$F_d(J_1, J_2) = \left(\prod_{j\in J_1} x_j\right)\left(\prod_{j\in J_2}(1-x_j)\right), \text{ where}$$

$$J_1, J_2 \subseteq [n],\ J_1 \cap J_2 = \emptyset \text{ and } |J_1 \cup J_2| = d.$$

$$\begin{tikzcd} {} \arrow[d, "d=1"] \\ {} \end{tikzcd}$$

$$x_1,\ x_2,\ (1-x_1) \text{ and } (1-x_2)$$

$$\begin{tikzcd} {} \arrow[d, "x_i^2 = x_i,\ x_1x_2 = w_{12}", "\times\ 2x_1 + 2x_2 \geq 1"'] \\ {} \end{tikzcd}$$

$$\boxed{\begin{array}{c} x_1 + 2w_{12} \geq 0, \\ x_2 + 2w_{12} \geq 0, \\ x_1 + 2x_2 - 2w_{12} \geq 1, \\ 2x_1 + x_2 - 2w_{12} \geq 1. \end{array}} \quad (1)$$

$$F_2(J_1, J_2) \geq 0$$

$$\begin{tikzcd} {} \arrow[d, "x_i^2 = x_i,\ x_1x_2 = w_{12}"] \\ {} \end{tikzcd}$$

$$\boxed{\begin{array}{c} w_{12} \geq 0, \\ x_1 - w_{12} \geq 0, \\ x_2 - w_{12} \geq 0, \\ 1 - x_1 - x_2 + w_{12} \geq 0. \end{array}} \quad (2)$$

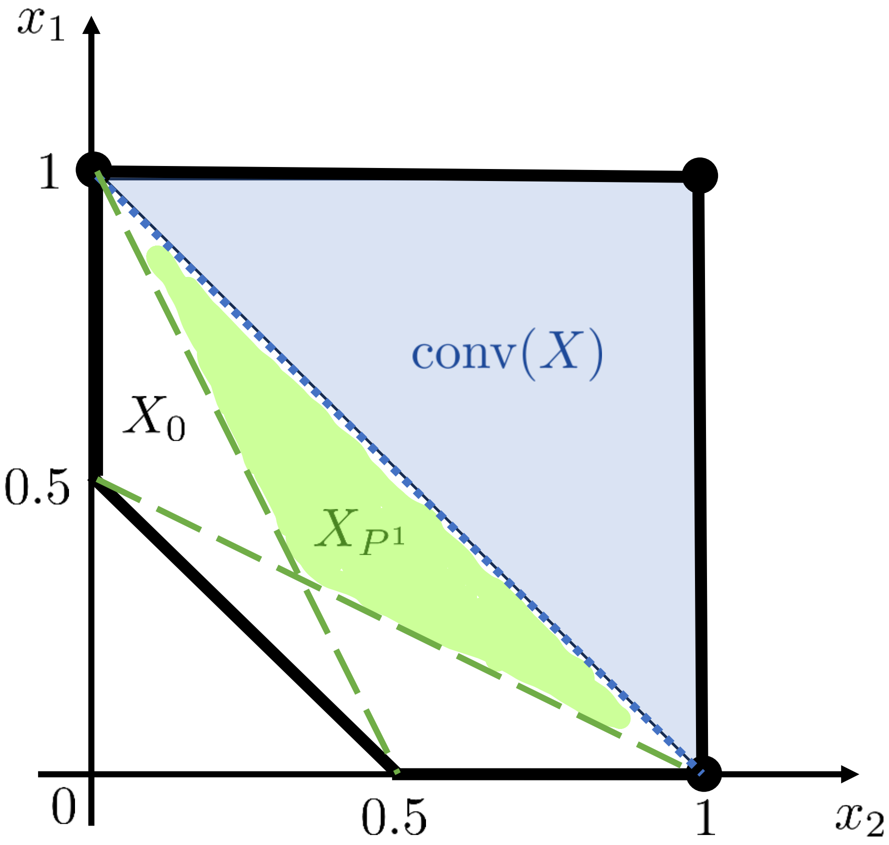
Sherali-Adams linearization Example

$$X_1 = \{(x_1, x_2, w_{12}) \in \mathbb{R}^3 \mid \text{constraints (1) and (2) hold}\}$$

$$\big\downarrow$$

projection

$$X_{P^1} = \{(x_1, x_2) \in \mathbb{R}^2 \mid 2x_1 + x_2 \geq 1,\ x_1 + 2x_2 \geq 1,\ x_1 \leq 1,\ x_2 \leq 1\}$$



Sherali-Adams linearization of Q-MWIS

$$\max_{x \in \{0,1\}^{n+m}} \sum_{i=1}^{n} c_i x_i + \sum_{ik \in N_Z} c_{ik} x_i x_k$$

$$\sum_{i \in K'_j} x_i = 1,\ \forall j \in [m] \qquad x_k^2 = x_k,\ \forall k \in [n+m]$$

$$\times$$

Polynomials of order $d = 1$

$$\forall i \in [n]:\ x_i,\ (1 - x_i)$$

$$\Longrightarrow$$

$$\forall j \in [m], k \in [n+m] : \begin{cases} \left(\sum_{i \in K'_j} x_i x_k \right) - x_k = 0, & \text{if } k \notin K'_j, \\ \sum_{i \in K'_j \setminus \{k\}} x_i x_k = 0, & \text{if } k \in K'_j \end{cases}$$

$$\forall j \in [m] : \sum_{i \in K'_j} x_i = 1$$

Sherali-Adams linearization of Q-MWIS

Non-negativity constraints $F_2(J_1, J_2) \geq 0$

$$\begin{gathered}\forall i, k \in [n+m], \text{ with } i \neq k : \\ x_i x_k \geq 0, \\ x_i(1 - x_k) \geq 0, \\ (1 - x_i)(1 - x_k) \geq 0,\end{gathered}$$

Sherali-Adams linearization of Q-MWIS

In total, after setting $x_i x_k = w_{ik},\ \forall ik \in [[n+m]]^2 :$

$$X_1 = \left\{ (x,w) \middle| \forall j \in [m], k \in [n+m] \text{ and } k \notin K'_j : \right.$$

$$\left(\sum_{\substack{s \in K'_j \\ s<k}} w_{sk} + \sum_{\substack{t \in K'_j \\ t>k}} w_{kt} \right) - x_k = 0;$$

$$\forall j \in [m], k \in [n+m] \text{ and } k \in K'_j :$$

$$\sum_{\substack{s \in K'_j \setminus \{k\} \\ s<k}} w_{sk} + \sum_{\substack{t \in K'_j \setminus \{k\} \\ t>k}} w_{kt} = 0;$$

$$\forall j \in [m] : \quad \sum_{i \in K'_j} x_i = 1;$$

$$\text{Additionally, } \forall ik \in [[n+m]]^2 :$$

$$w_{ik} \geq 0,$$

$$\boxed{\begin{aligned} & x_i - w_{ik} \geq 0, \\ & x_k - w_{ik} \geq 0, \\ & \left. w_{ik} - x_i - x_k + 1 \geq 0 \right\}. \end{aligned}}$$

Sherali-Adams linearization of Q-MWIS

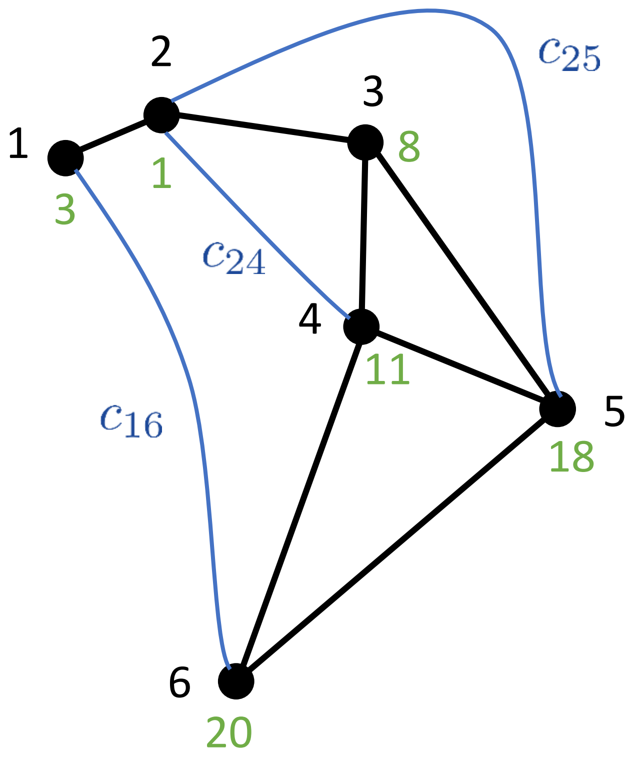
Linearized Q-MWIS problem with Sherali-Adams method:

$$\max_{\substack{x\in\{0,1\}^{n+m}\\ w\in\{0,1\}^{\frac{1}{2}(n+m-1)(n+m)}}} \sum_{i=1}^{n} c_i x_i + \sum_{ik\in N_Z} c_{ik} w_{ik}$$

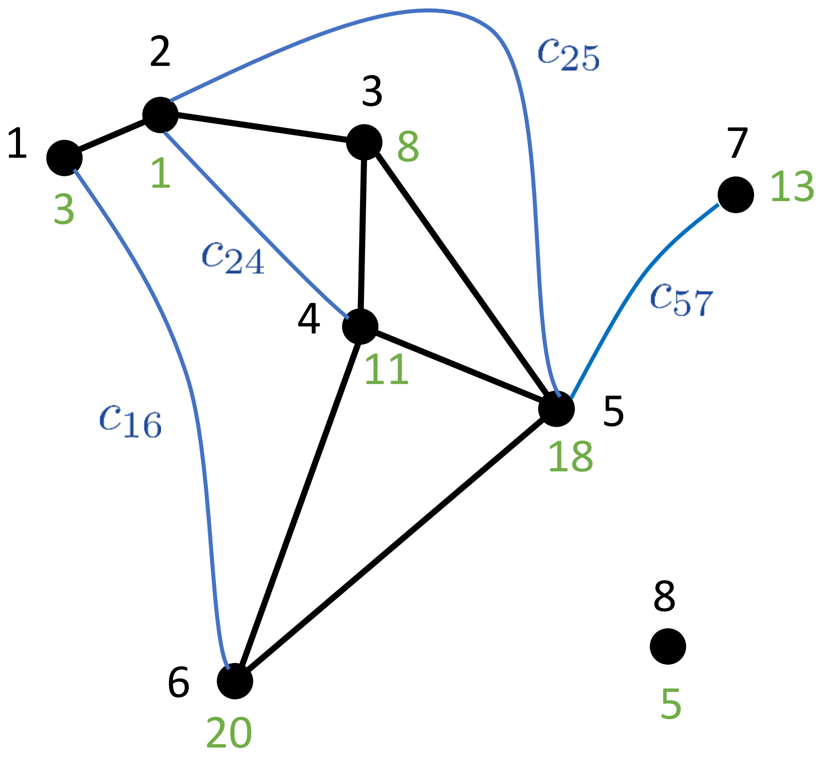
$$\text{s.t. } (x,w)\in X_1.$$

Q-MWIS standard form

Definition:



in standard form



not in standard form

Redundant constraints

Theorem: If Q-MWIS problem in standard form …

and $\forall i \in [n+m]:$

$$x_i \geq 0$$

$\Rightarrow \forall ik \in [[n+m]]^2:$

$$\cancel{\begin{aligned} x_i - w_{ik} &\geq 0, \\ x_k - w_{ik} &\geq 0, \\ w_{ik} - x_i - x_k + 1 &\geq 0 \end{aligned}}$$

Concise Linearization

$$X_1' = \Bigg\{ (x,w) \,\Bigg|\, \forall j \in [m], k \in [n+m] \text{ and } k \notin K_j' :$$

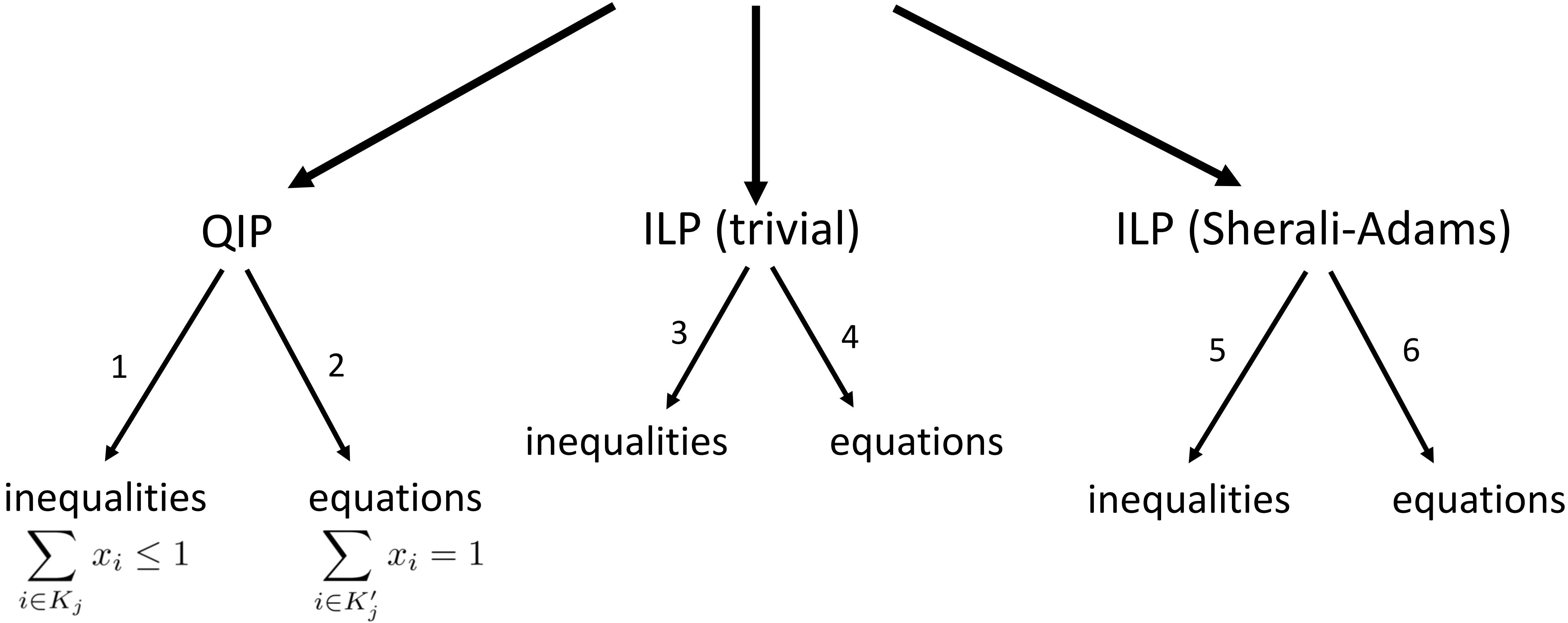
$$\left(\sum_{\substack{s\in K_j' \\ s<k}} w_{sk} + \sum_{\substack{t\in K_j' \\ t>k}} w_{kt} \right) - x_k = 0;$$

$$\Bigg\} \quad \forall ik \in [[n+m]]^2 : \quad \text{If } \exists j \in [m] : \; i,k \in K_j' \Rightarrow w_{ik} = 0.$$

Tighter relaxation, fewer constraints than trivial linearization.

Performance testing candidates

Which problem formulation performs best?



QIP

1

inequalities

$$\sum_{i \in K_j} x_i \leq 1$$

2

equations

$$\sum_{i \in K'_j} x_i = 1$$

ILP (trivial)

3

inequalities

4

equations

ILP (Sherali-Adams)

5

inequalities

6

equations

Performance test method

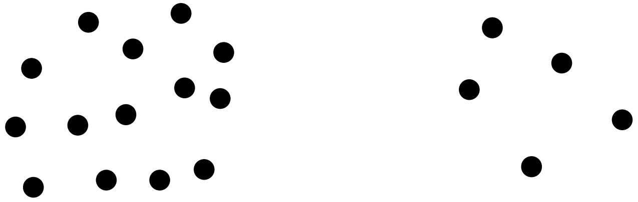
Generate sets of Q-MWIS problem instances of varying size/structure

↓

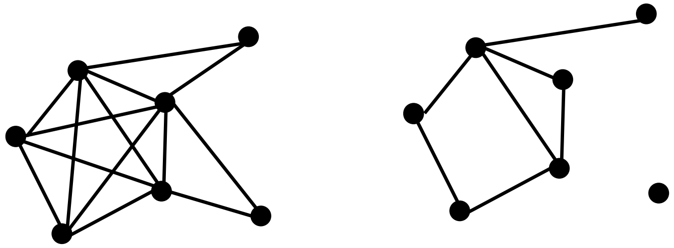
Solve each instance with different problem formulations using Gurobi solver via Python API

↓

Compare average solution time/MIP gap for every problem set



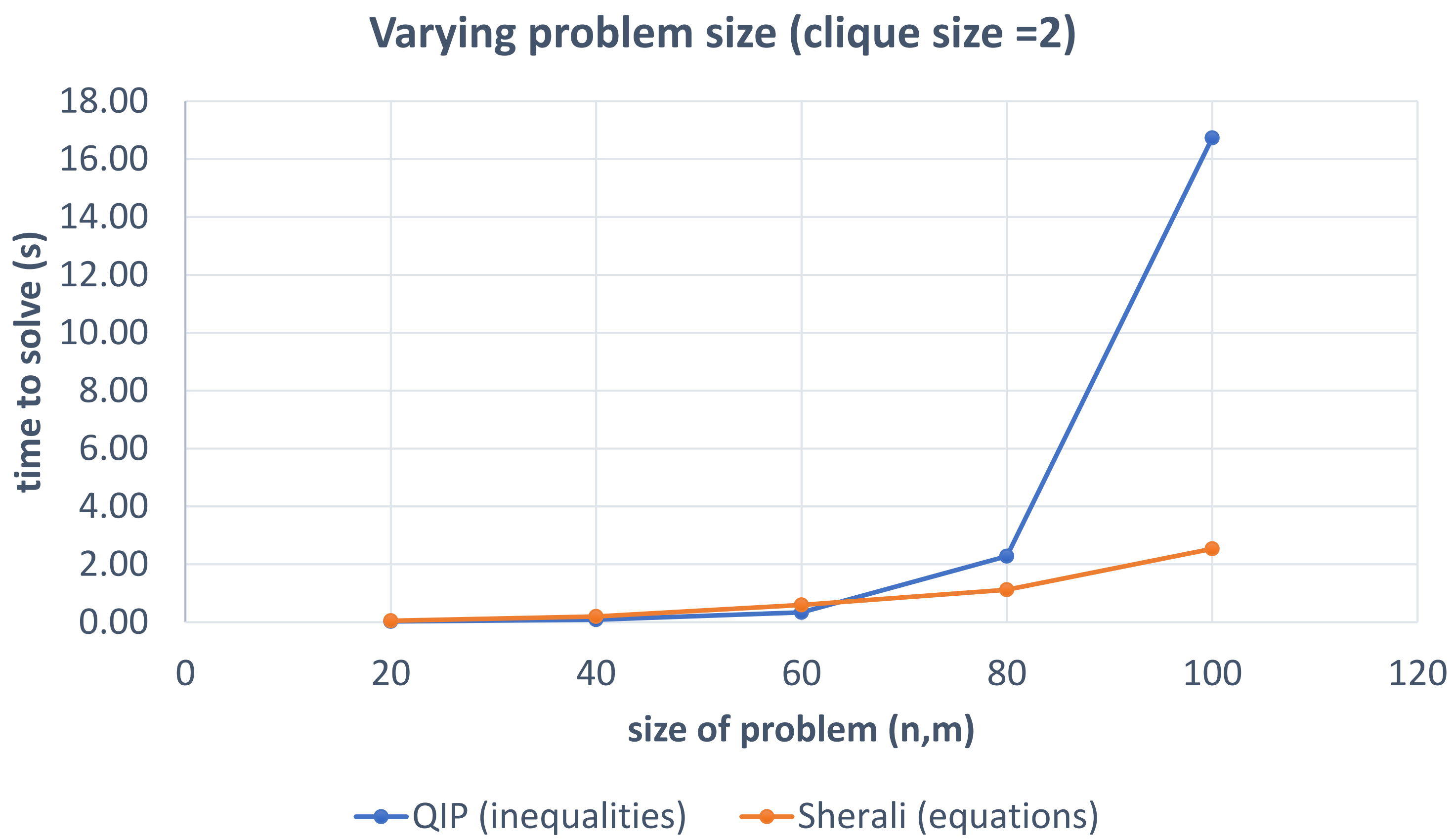
large/small n



large/small cliques/conflict sets

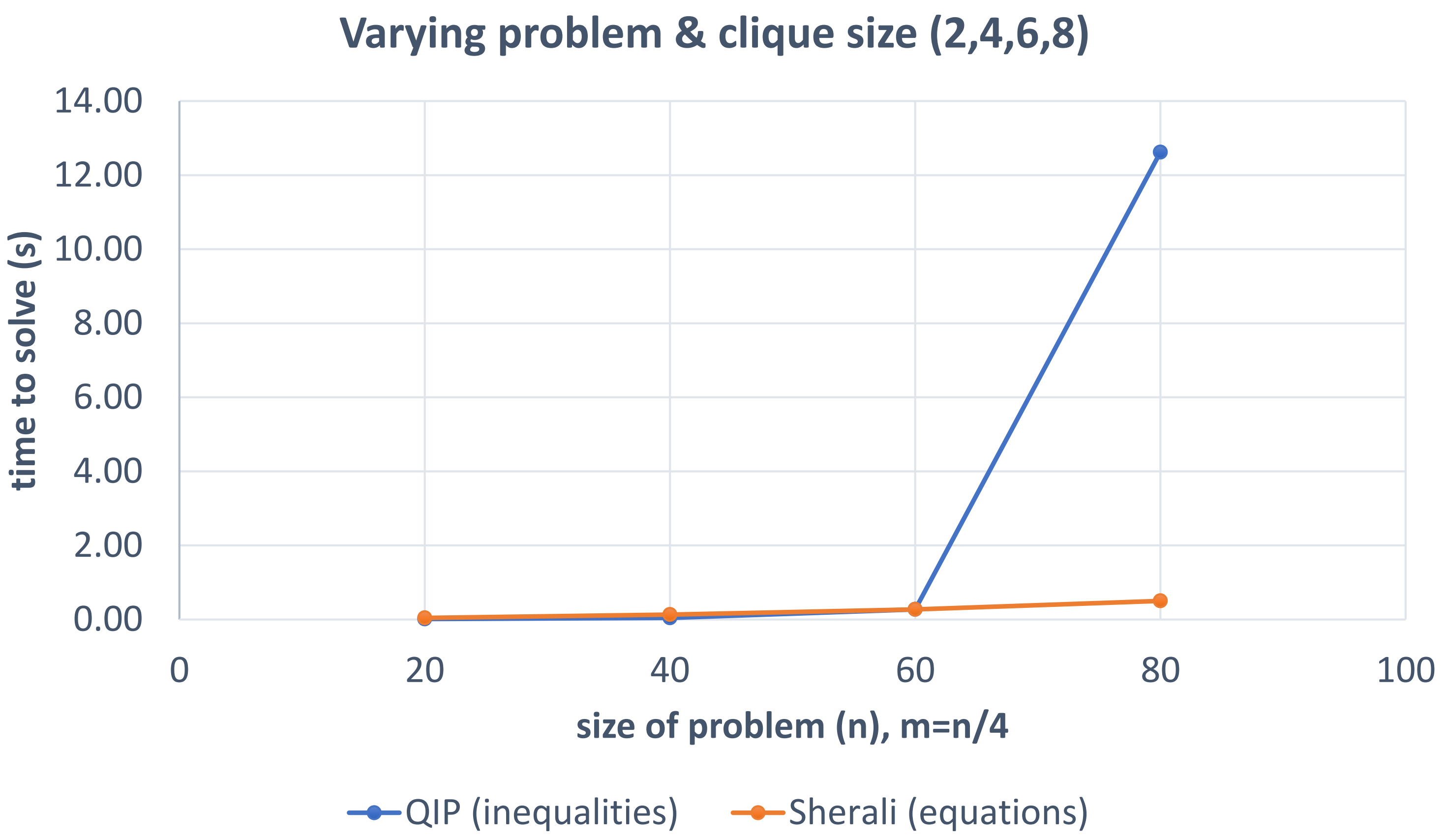
dense/sparse pairwise costs

Performance test results



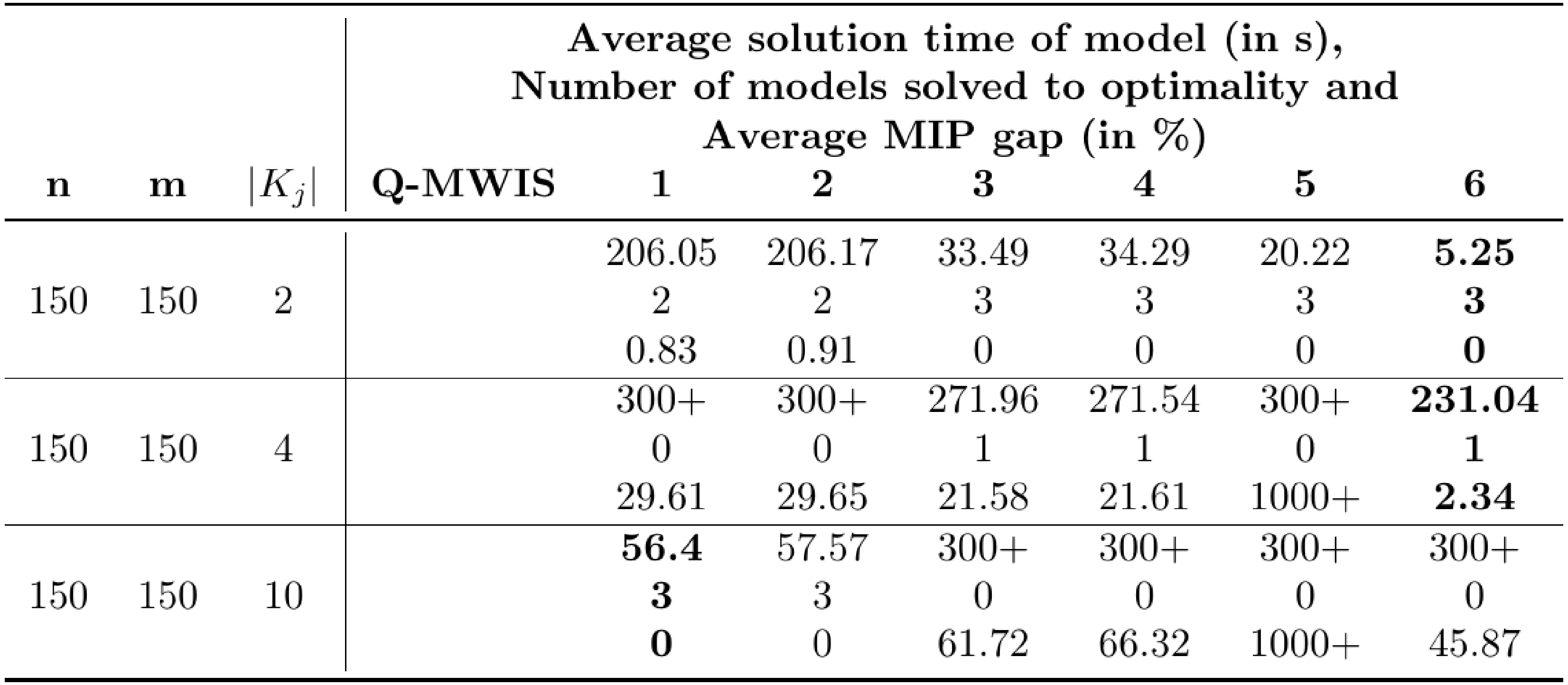
larger problems
=> linearizations better

Performance test results



bigger cliques
=> Sherali-Adams better

Performance test results



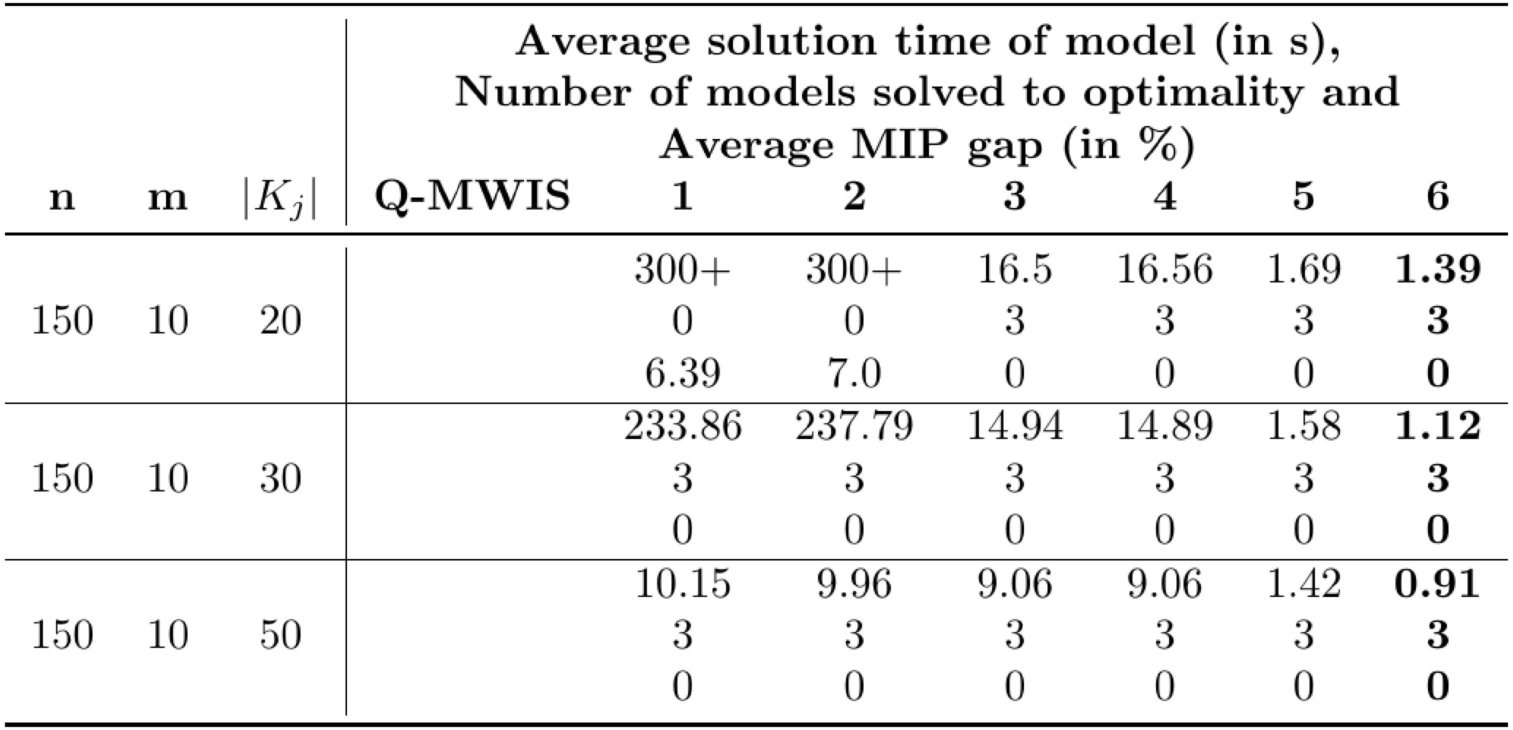
n	m	$\lvert K_j \rvert$	Q-MWIS	1	2	3	4	5	6
			Average solution time of model (in s), Number of models solved to optimality and Average MIP gap (in %)						
150	150	2		206.05 2 0.83	206.17 2 0.91	33.49 3 0	34.29 3 0	20.22 3 0	**5.25** **3** **0**
150	150	4		300+ 0 29.61	300+ 0 29.65	271.96 1 21.58	271.54 1 21.61	300+ 0 1000+	**231.04** **1** **2.34**
150	150	10		**56.4** **3** **0**	57.57 3 0	300+ 0 61.72	300+ 0 66.32	300+ 0 1000+	300+ 0 45.87

?

Problem too large
=> Gurobi gets stuck,
solving LP relaxation

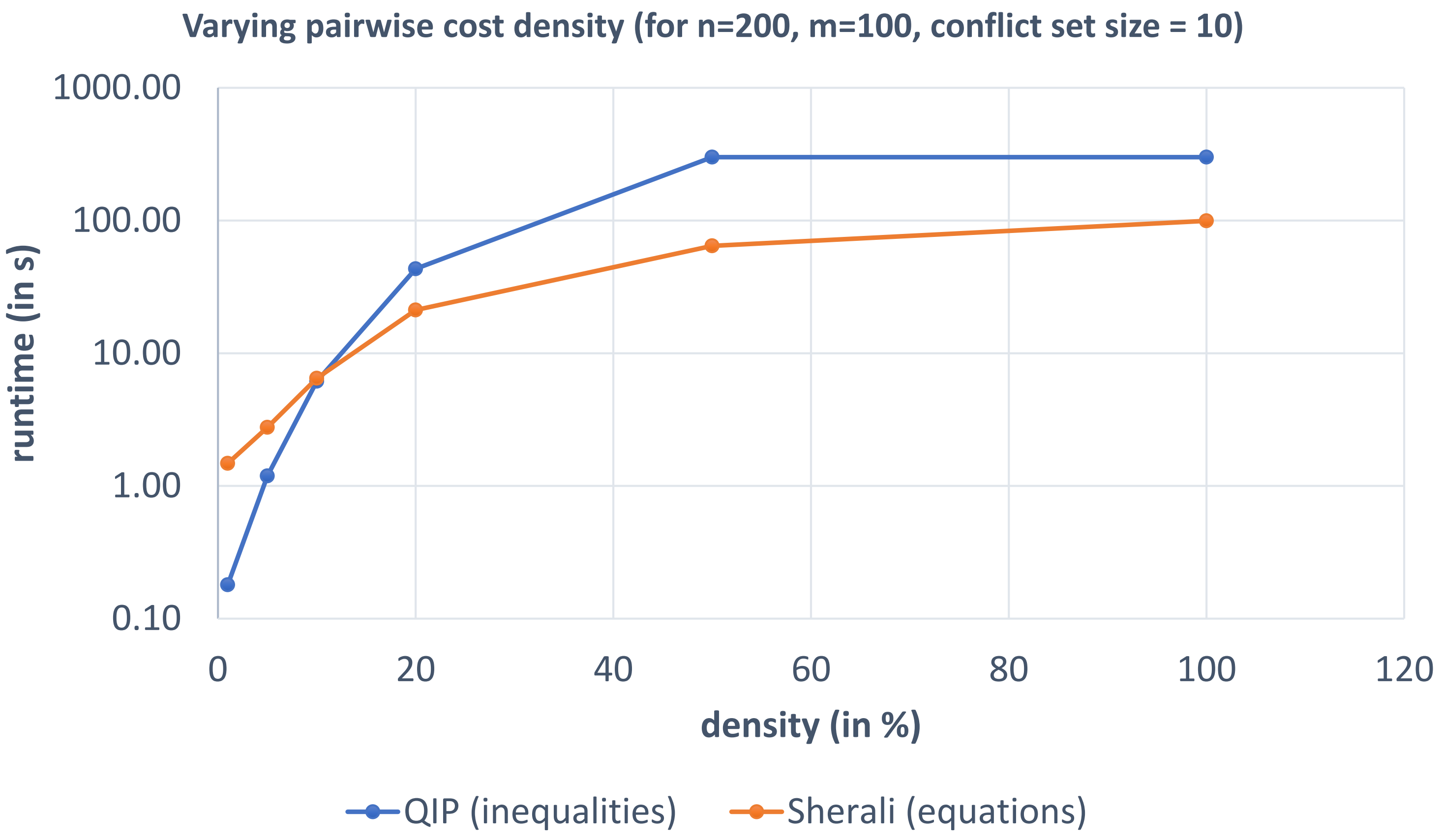
Performance test results

… doesn‘t happen with fewer + larger conflict sets



			Average solution time of model (in s), Number of models solved to optimality and Average MIP gap (in %)						
n	**m**	$\lvert K_j \rvert$	**Q-MWIS**	**1**	**2**	**3**	**4**	**5**	**6**
150	10	20		300+ 0 6.39	300+ 0 7.0	16.5 3 0	16.56 3 0	1.69 3 0	**1.39** **3** **0**
150	10	30		233.86 3 0	237.79 3 0	14.94 3 0	14.89 3 0	1.58 3 0	**1.12** **3** **0**
150	10	50		10.15 3 0	9.96 3 0	9.06 3 0	9.06 3 0	1.42 3 0	**0.91** **3** **0**

Performance test results



denser pairwise costs
=> Sherali-Adams better

Performance test results

Sherali-Adams with equality constraints > other linearizations …

… but implementation has room for improvement in large problems

Summary